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ANOTHER INTERPRETATION OF THE FUNDAMENTAL GAUGE-VECTOR OF WEYL'S THEORY OF RELATIVITY

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1. In the geometry of paths as developed by Professor Veblen and myself in a number of papers in volumes eight and nine of these Proceedings, the idea is that the paths are a generalization of straight lines in euclidean space. We take the equations of the paths in the form

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0, \tag{1}$$

where s is a parameter peculiar to each path and $\Gamma^i_{\alpha\beta}$ $(=\Gamma^i_{\beta\alpha})$ are functions of the x's. Now I make the assumption that *physical phenomena manifest themselves in paths in a space-time continuum of four dimensions and that the functions $\Gamma^i_{\alpha\beta}$ are determined by the character of the phenomena.* In this note I apply this idea to the case of electro-magnetic phenomena as developed in the general theory of relativity, and the results raise the question whether Weyl, and later Eddington, are justified in the assumption that the fundamental vector introduced by Weyl in his gauging system is the electro-magnetic potential of the field.

2. Suppose that the fundamental quadratic form of the space-time continuum is written in the general form

$$ds^2 = g_{ij}dx^i dx^j \qquad (g_{ij} = g_{ji}) \tag{2}$$

Let K_α and K^β denote the covariant and contravariant components of the electro-magnetic potential in a general system of coördinates, x^i. As usual we put

$$F_{\mu\nu} = \frac{\partial K_\mu}{\partial x^\nu} - \frac{\partial K_\nu}{\partial x^\mu}, \tag{3}$$

and

$$F^{\mu}_{\nu} = g^{\mu\alpha}F_{\alpha\nu}, \qquad F^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}, \tag{4}$$

where $g^{\mu\alpha}$ is the cofactor of $g_{\mu\alpha}$ in the determinant divided by

$$g = |g_{\mu\nu}| \tag{5}$$

From (3) follows the second set of Maxwell equations

$$\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} + \frac{\partial F_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial F_{\sigma\mu}}{\partial x^{\nu}} = 0, \tag{6}$$

and the first set in tensor form are

$$F^{\mu\nu}_{\nu} = J^{\mu} \tag{7}$$

where the left-hand member is the sum for ν of the covariant derivatives of $F^{\mu\nu}$, and J^{μ} are the contravariant components of the charge-current vector in general coördinates; thus

$$J^{\mu} = \rho_0 \frac{dx^{\mu}}{ds}, \tag{8}$$

where ρ_0 is the proper-density of the charge.[1]

The equations of motion, in tensor form, of an electric charge are

$$\mu\left(\frac{d^2x^i}{ds^2} + \left\{\begin{matrix}\alpha\beta\\ i\end{matrix}\right\}\frac{dx^{\alpha}}{ds}\frac{dx^{\beta}}{ds}\right) = -F^i_{\nu}J^{\nu}, \tag{9}$$

where μ is the mass-density and $\left\{\begin{matrix}\alpha\beta\\ i\end{matrix}\right\}$ denotes the Christoffel symbols of the second kind with respect to (2).[2]

3. If we put

$$\varphi^i = -\frac{1}{\mu}F^i_{\nu}J^{\nu} = -\frac{1}{T}F^i_{\nu}J^{\nu}, \tag{10}$$

where, in consequence of (2),

$$T = \mu g_{\alpha\beta}\frac{dx^{\alpha}}{ds}\frac{dx^{\beta}}{ds} = \mu, \tag{11}$$

equations (9) may be written

$$\frac{d^2x^i}{ds^2} + \left(\left\{\begin{matrix}\alpha\beta\\ i\end{matrix}\right\} - g_{\alpha\beta}\varphi^i\right)\frac{dx^{\alpha}}{ds}\frac{dx^{\beta}}{ds} = 0. \tag{12}$$

These equations are of the form (1), where

$$\Gamma^i_{\alpha\beta} = \left\{ \begin{matrix} \alpha\beta \\ i \end{matrix} \right\} - g_{\alpha\beta}\varphi^i. \tag{13}$$

Elsewhere,[3] I have shown that the same paths are defined by

$$\frac{d^2x^i}{ds'^2} + \Gamma'^i_{\alpha\beta}\frac{dx^\alpha}{ds'}\frac{dx^\beta}{ds'} = 0, \tag{14}$$

if we take

$$\Gamma'^i_{\alpha\beta} = \delta^i_\alpha\varphi_\beta + \delta^i_\beta\varphi_\alpha, \qquad \frac{d^2s}{ds'^2} = -2\varphi_\alpha\frac{dx^\alpha}{ds}\left(\frac{ds}{ds'}\right)^2. \tag{15}$$

From (8), (10) and the skew-symmetric character of $F_{\mu\nu}$, we have

$$\varphi_\alpha\frac{dx^\alpha}{ds} = 0, \tag{16}$$

and hence from (15) we see that s and s' can be taken as the same. From (13) and (15) it follows that the functions Γ for the paths may be given the form

$$\Gamma^i_{\alpha\beta} = \left\{ \begin{matrix} \alpha\beta \\ i \end{matrix} \right\} + \delta^i_\alpha\varphi_\beta + \delta^i_\beta\varphi_\alpha - g_{\alpha\beta}\varphi^i. \tag{17}$$

The expressions (17) are those given by Weyl[4] for the coefficients of the affine connection of a manifold whose character is determined directly by gravitation and electro-magnetism. Weyl assumes that the vector φ_α in (17) is the electro-magnetic potential. From the foregoing results it would seem that φ_α is not the electro-magnetic potential but is functionally connected with it, K_α, as given by (10), (3) and (7). There are other considerations which point to this view.

4. In accordance with the Weyl[5] geometry a new gauge system is introduced by altering the length of the standard at each point in the ratio $\lambda^{1/2}$, where λ is an arbitrary point function. If the standard is decreased in the ratio $\lambda^{1/2}$, a length will be increased in this ratio and for the new system

$$ds'^2 = \lambda ds^2, \qquad g'_{\alpha\beta} = \lambda g_{\alpha\beta}, \tag{18}$$

the coördinate system being independent of the gauge.

If the components of a tensor in the new gauge system are λ^e times those in the old system, the tensor is said to be of weight e. Thus from (18) it follows that $g_{\alpha\beta}$ is of weight one, and from (12) that φ^i is of weight

minus one, and consequently $\varphi_\mu\ (= g_{\mu i}\varphi^i)$ is of weight zero. Furthermore it follows from (3), (4) and (7) that if K_α is taken of weight one, then φ^i is of weight minus one from (10), which is in accordance with (12).

Pauli[6] pointed out that the Einstein equations

$$R_{ik} - \frac{1}{2} g_{ik} R = - kT_{ik} \tag{19}$$

are inconsistent, when it is assumed, as Weyl has done, that the convariant components K_α of the electro-magnetic potential are of weight zero. In fact, on this hypothesis, the left-hand and right-hand members of (19) are respectively of weights zero and minus one. Jüttner[7] called attention to the same difficulties in the case of the equations

$$R_{ik} - \frac{1}{4} g_{ik} R = - kS_{ik}, \tag{20}$$

where

$$S_{ik} = \frac{1}{4} g_{ik} F_{\alpha\beta} F^{\alpha\beta} - F_{i\alpha} F_{k\beta} g^{\alpha\beta}.$$

He tried to overcome the inconsistencies by changing the form of these equations, while still holding to the hypothesis that K_α is of weight zero.

Mr. A. Bramley of the Department of Physics of Princeton University has shown, in a paper to be offered to the *Philosophical Magazine*, that equations (19) and (20) are consistent, if the weight of K_α is taken to be one, and if K_α is not supposed to be the fundamental vector φ_α of the gauging system, but functionally related to it in such a way that φ_α is of weight zero. Equations (10) are consistent with the hypothesis made by Bramley. In view of the foregoing considerations it is a question whether Weyl's assumption that the gauging vector φ_i is the electro-magnetic potential should be rejected, and the relation (10) be adopted to give the relation between φ_α and the electro-magnetic potential K_α.

[1] Eddington, *The Mathematical Theory of Relativity*, Cambridge, 1923, pp. 173, 190.

[2] Eddington, *l. c.*, p. 190.

[3] These PROCEEDINGS, **8**, 1922 (234); also Veblen, *Ibid.* (347).

[4] Cf. Weyl, *Space, Time and Matter*, English translation, pp. 125, 284; also Eddington, *l. c.*, pp. 202, 203.

[5] *L. c.*, p. 127; also, Eddington, *l. c.*, pp. 200, 203, Eddington shows that (17) is invariant for gauge-transformations, and for this reason (17) is used in place of (13). As shown by (14), (15) and (16) the forms (17) and (13) lead to the same paths.

[6] *Encycl. Math. Wiss.*, vol. V_2, part 4, p. 767.

[7] *Math. Ann.*, **87**, 1922 (282).

TENSOR ANALYSIS WITHOUT COÖRDINATES

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From the time when the theory of curved space was introduced by Riemann and Christoffel there were many attempts to simplify its foundations but it seems to me that in spite of very important contributions of Ricci, Levi Civita, Weyl, Schouten and others the theory was not yet reduced to its simplest form. Since the classical theory is an immediate generalization of the theory of surfaces it seems expedient to deal exhaustingly with this theory first; this paper indicates how it can be done with the use of a few notions of vector analysis in a simple way; the complications with which the ordinary tensor analysis is encumbered are then shown to arise from the method of representation. In another paper the author will deal with generalizations of the theory and its relations to the foundations of geometry and to physics.

Preliminary remarks and notations.—We denote points with capital Roman letters, vectors with small Roman letters and we use Greek letters for scalars (real numbers). We adopt the usual definitions for the addition of vectors, for the multiplication of a vector by a scalar and the (scalar) multiplication of two vectors x and y which we denote $x.y$ or xy. We denote by $B—A$ the vector which leads from A to B and by $A + h$ the position of the endpoint of the vector h when its initial point is made to coincide with A.

All the functions which we consider are supposed to be continuous. A scalar function $\varphi(x)$ or a vector function $f(x)$ is called *linear* if it satisfies the condition

$$\varphi(x + y) = \varphi(x) + \varphi(y) \quad \text{or} \quad f(x + y) = f(x) + f(y).$$

A scalar function of several vector arguments $\varphi(x,y...)$ which is linear in each of them is called a *tensor* and the number of arguments the *rank* of the tensor. A tensor of the first rank $\varphi(x)$ can be represented as the product $c.x$ of the argument x with a constant vector c; examples of a tensor of the second rank are $x.y$ and $(ax)(ay)$; a tensor of the second rank can be represented as the product of one of its arguments say x with a linear vector function of the other

$$\varphi(x, y) = x.s(y) \tag{1}$$

We call a function $\varphi(A,x,y..)$ with one point argument A and several vector arguments $x,y..$ a *tensorfield* if for every fixed value of A it is a ten-

sor, and the number of its vector arguments we call its *rank*. A scalar function of a point $\varphi(A)$ we call a tensorfield of the *rank zero*. The *differential* of a tensorfield of the rank zero $\varphi(A)$ is the tensorfield $\varphi(A,h)$ of the rank one which is given by

$$\varphi(A, h) = \lim_{\lambda \to 0} \frac{\varphi(B) - \varphi(A)}{\lambda}, \tag{2}$$

with the condition $B = A + \lambda h$. The differential of a field $\varphi(A,x,y...)$ of the rank r is the field $\varphi(A,x,y...h)$ of the rank $r + 1$ which is obtained in the following way: we give to the vector arguments $x,y..$ in the original tensorfield fixed values and we thus have a field of rank zero; we find the differential of this field by the rule (2); then we restore the variability to the arguments $x,y...$.

We consider a curve as a function $C(\tau)$ the argument τ of which is a scalar—the length of the arc—and the values of which are points in space—points of the curve. The expression

$$\lim_{\rho \to \tau} \frac{B - A}{\rho - \tau} \text{ with } B = C(\rho), \quad A = C(\tau) \tag{3}$$

defines a unit vector $t = t(\tau)$ which is tangent to the curve in the point A. The expression

$$\lim_{\rho \to \tau} \frac{t_B - t_A}{\rho - \tau} \text{ with } t_B = t(\rho), \quad t_A = t(\tau) \tag{4}$$

defines a vector $t = t(\tau)$ which belongs to the principal normal and the length of which is the curvature of the curve in the point A.

Elements of the theory of surfaces.—We will consider now a surface having a tangent plane in every point. The totality of vectors belonging to the tangent plane in A which vectors will be denoted by x_A, y_A... we will call the *bundle* A. The orthogonal projection of the point B on the tangent plane in A will be designated by B_A. The unit normal vectors in the points P, A, B... will be designated by p, a, b... . We have relations $x_A.a = 0$, etc.

We will consider tensorfields $\varphi(A,x_A,y_A...)$ *belonging* to the surface: their point arguments will be only points of the surface and their vector arguments—vectors of the bundle corresponding to the point argument. The simplest example of such a tensorfield is obtained when we take in every point of the surface the product of two variable vectors of the corresponding bundle

$$\epsilon(A,x_A,y_A) = x_A.y_A. \tag{5}$$

Let $\delta(B_A)$ denote the distance between the points B and B_A. This

is a tensorfield (not belonging to the surface) of the rank zero defined in the tangent plane of the point A. Let $\delta(B_A,h_A)$ be its differential and $\delta(B_A,h_A,k_A)$ the differential of this differential; then $\delta(A,h_A,k_A)$ is a tensorfield belonging to the surface. We know [comp. (1)] that it can be represented in the form $h_A.s(k_A)$ where s is a linear vector function in the bundle A. We can prove that

$$\lim_{\lambda \to 0} \frac{b - a}{\lambda} = - s(h_A) \text{ with } B_A = A + \lambda h_A. \tag{6}$$

The two tensorfields of the second rank ϵ and δ correspond to the first and second differential forms of Gauss. If a curve is given on the surface its curvature in the point A is given by

$$\frac{\delta(A,t_A,t_A)}{n.a} = \frac{t_A.s(t_A)}{n.a}, \tag{7}$$

where n is the unit vector of the principal normal of the curve. In this expression are contained both the Euler and the Meusnier theorems.

Differentiation on the surface.—The definition of the differential of a tensorfield of the rank zero given by (2) has to be but slightly modified for the case of a field belonging to a surface: instead of condition $B = A + \lambda h$ we have now to make $B_A = A + \lambda h_A$. But if we have a tensorfield of a higher rank $\varphi(A,x_A,y_A...)$ we cannot consider it as a field of the rank zero by giving to the vector arguments fixed values—a device which permitted us before to define the differential of a tensorfield of an arbitrary rank with other words if we take the expression

$$\lim_{\lambda \to 0} \frac{\varphi(B,x_B,y_B...) - \varphi(A,x_A,y_A...)}{\lambda}$$

we cannot put here $x_B = x_A$, $y_B = y_B...$ because equality between vectors of different bundles has no meaning; we therefore introduce the condition that the projection of x_B on the plane tangent in A should be equal to x_A; that seems to be the most natural thing to do and its success shows that it is the right thing. The projection of x_B on the plane tangent in A which we will denote by $x_{B.A}$ is given by the formula

$$x_{B.A} = x_B - a(x_B.a), \tag{8}$$

the vector $x_{A.B}$ whose projection on the tangent plane in A is equal to x_A is given by

$$x_{A.B} = x_A - a.\frac{x_A.b}{a.b}. \tag{9}$$

The definition of a differential can be given now in the form

$$\varphi(A,x_A,y_A..,h_A) = \lim_{\lambda \to 0} \frac{\varphi(B,x_{A\cdot B},y_{A\cdot B}...) - \varphi(A,x_A,y_A,...)}{\lambda}, \qquad B_A = A + \lambda h_A. \tag{10}$$

If we have a curve on the surface let us consider as the derivative on the surface of t [comp. (4)] the expression

$$t' = \lim_{\rho \to \tau} \frac{t_{B.A} - t_A}{\rho - \tau}.$$

We find easily

$$t' = \dot{t} - a(\dot{t}.a).$$

The condition $t' = 0$ gives us the geodesics; the length of the vector t' is the geodesic curvature of the curve.

If we put $B_A = A + \lambda h_A$, $C_A = A + \mu k_A$ it can be shown that

$$\lim_{\substack{\lambda \to 0 \\ \mu \to 0}} \frac{x_{A.B.C.A} - x_{A.C.B.A}}{\lambda.\mu} = s(k_A)[s(h_A.).x_A] - s(h_A)[s(k_A).x_A] = \gamma.\{k(xh) - h(xk)\}. \tag{11}$$

This is a trilinear vector function; multiplying it by a vector y we get a quadrilinear scalar function—a tensor of the rank four which is the Riemann tensor.

Representations.—We have thus obtained everything that is essential in the elements of the theory of surfaces in a form capable of immediate generalization to n dimensions without having introduced the notions which have no intrinsic significance such as transformations, covariant and contravariant quantities, the fundamental tensor g_{ij}, the symbols with three indices, etc. All these things do not belong to the object of our study—to the surface or the n-space—they are introduced by the method which is applied, by the representation which is used. We shall now briefly outline how the usual representation introduces these things.

We take a fixed point P and the tangent plane Π in this point. We *represent* a point of the surface by its projection on Π and for a vector x_A we use two kinds of representation: the lower $x_{A.P}$ [comp. (8)] and the upper $x_{A.P}$ [comp. (9)]. Between these two representations there exist two relations

$$x_{A.P\cdot A} = x_A \text{ and } x_{A.P}.y_{A\cdot P} = x_A.y_A. \tag{12}$$

If we introduce Cartesian coördinates in the plane Π and write x^1,x^2 for the components of a vector of this plane when we use it as an upper rep-

resentation, and x_1,x_2 when we use it as a lower representation, we have contravariant and covariant components of a vector. The representation of a tensor appears as a multilinear form the variables of which are these components of vectors; we get different coefficients for a form representing a given tensor depending on whether we use co- or contravariant components for the vectors; these coefficients are contra- or covariant or mixed components of tensors. The absolute differentiation arises out of the question: given a representation of a tensorfield to find the representation of its differential. The g_{ij} are a representation of the simple tensor $\epsilon(A, x_A, y_A) = x_A . y_A$ and the relations (12) account for all the rules for passing from covariant to contravariant components, etc. The tensor ϵ itself is a most innocent thing: it has the same properties in all points—it is the same tensor indeed in all points, in accordance with which its differential is zero. The important part played by its representation g_{ij} in the usual theory is based on the fact that the properties of the surface are involved in the method of representation.

The question arises now whether there can be devised other representations which do not introduce these complications.

GEOMETRIC ASPECTS OF THE ABELIAN MODULAR FUNCTIONS OF GENUS FOUR (III)[11]

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12. *The form* $\binom{112}{112}$.—This form, written symbolically as $(\alpha x)^2(b\tau)(\beta t)$ where t, τ are digredient binary variables and x is a point in S_2, has $2.2.6-1-2.3-8 = 9$ absolute projective constants. The locus of points x for which the bilinear form in t, τ factors is the quartic curve C^4, $(\alpha x)^2(\alpha' x)^2(bb')(\beta\beta') = 0$. For fixed τ and variable t the pencil of conics determined by the form has four base-points on C^4 which form a quadruple p_τ; similarly for fixed t and variable τ there is defined on C^4 a quadruple q_t. The quadruples p_τ form a linear series g_4^1 on C^4; the quadruples q_t a series h_1^4; and these series are residual with respect to each other. Conversely a quadruple on a general quartic C^4 determines its linear series, its residual series of quadruples, and thereby projectively determines the original form. If for given t, τ the conic $(\alpha x)^2(b\tau)(\beta t) = 0$ has a double point x then, according to Wirtinger,[7] the locus of such points x, the vertices of the diagonal triangles of either system of co-residual quadruples, is a plane sextic of genus 4 whose special canonical series are determined by $(\alpha\alpha'\alpha'')^2\ (b\tau)$-

$(b'\tau)(b''\tau)(\beta t)(\beta' t)(\beta'' t) = 0$. Wirtinger obtains the equation of this sextic as the discriminant of the conic $(\alpha x)(\alpha x_1)(\alpha' x)(\alpha' x_1)(bb')(\beta\beta') = 0$ in variables x_1.

13. *The form* $\binom{112}{112}$ *as the equivalent of a Steiner surface and a quadric.*— Proceeding from the results above we observe that the conics defined by the form $\binom{112}{112}$ lie in a *web* which can be exhibited more symmetrically by replacing the bilinear combinations of t,τ by the coördinates of a point y in S_3. The form then becomes $(\alpha x)^2(ay) = 0$. To a point x of S_2 there corresponds a plane in S_3 which, as x runs over its S_2, envelops a Steiner quartic envelope whose reciprocal point locus is a Steiner four-nodal cubic surface. The effect of replacing the coördinates y by t,τ is to introduce into the S_3 a quadric I with generators t,τ. Then I cuts the Steiner cubic surface in a point sextic of genus 4. Moreover I has in common with the Steiner quartic envelope an octavic locus of planes of genus 3 and the tetrads of planes of this locus which contain the generators t,τ mark on the locus of genus 3 the two residual series of coresidual quadruples. From these considerations one may show that the projective peculiarity of the birationally general Wirtinger sextic, W, of genus 4 is that *its six nodes are the vertices of a four line.* The pencil of conics on this four line is the pencil apolar to the original web. When the plane is mapped by cubic curves on the six vertices it becomes a four-nodal cubic surface and the W-sextic maps into a general space sextic of genus 4 on a quadric I, while the quartic curve C^4 maps into a curve of order 12 on the Steiner cubic surface along which the tangent planes of the surface form the complete octavic intersection of the Steiner envelope and I. The quadric $(\alpha\alpha' u)(\alpha\alpha' v)(ay)(a'y) = 0$ meets the space sextic in the 12 points which are the maps of the points in which the lines u, v of the plane meet W whence $(\alpha\alpha' u)^2(ay)(a'y) = 0$ is, for variable u, a system of contact quadrics of the space sextic. Since each of these is associated with one of the 255 proper half periods of the allied theta functions of genus 4 we have the beautiful theorem:—*On the general space sextic of genus 4 there are precisely 255 four-nodal cubic surfaces each associated with one of the half periods of the corresponding theta functions.*

If u is one of the 28 double tangents of C^4 the contact quadric $(\alpha\alpha' u)^2(ay)(a'y) = 0$ is a contact cone of the space sextic which with the quadric I lies in a pencil of which one member is a pair of tritangent planes. The 28 pencils of this sort give rise to the so-called Steiner complex[12] of 28 pairs of tritangent planes. With respect to each of these 255 Steiner complexes the following theorems may be proved. (*a*) The 28 pairs of tritangent planes of a space sextic of genus 4 which form a Steiner complex are pairs of a Cremona involution in space with the tropes of the Steiner quartic envelope as F-planes. The 28 lines in which the pairs of planes meet are lines of a cubic complex.

The analogous theorem for the double tangents of a plane quartic is that the six pairs of double tangents of a Steiner complex meet in six points on a conic.

The nodal tetrahedron of the Steiner cubic surface has edges on the surface and therefore bisecant to the space sextic. Given a space sextic and such a bisecant tetrahedron the cubic surface can be reconstructed whence (*b*) For a space sextic of genus 4 there are 255 tetrahedra whose edges are bisecants of the curve. The 28 pairs of planes of a Steiner complex cut an edge of the nodal tetrahedron of the allied Steiner cubic in pairs of points of an involution determined by the pair of nodes on that edge and by the pair of crossings of the sextic on that edge.

The Wirtinger scheme in the plane is determined by the general sextic surface with four-fold points and a plane section. For the cubic Cremona involution with F-points at the four nodes of a Steiner cubic transforms the Steiner cubic into a plane and the quadric I into such a sextic surface.

If C_1, C_2 are two Steiner cubic surfaces on the space sextic then $C_1 + kC_2 = \pi I$. Hence C_1 and C_2 cut the plane π in the same cubic curve C_{12}. Moreover on C_{12} in π the nodal tetrahedra of C_1 and C_2 cut out two inscribed four lines of C_{12}. Two cases are possible according as these two four lines in C_{12} correspond to the same or to different half periods on the elliptic cubic C_{12}. Either of these cases can occur. For given the plane, the cubic curve, and two inscribed four lines belonging to the same or to different systems, two Steiner cubic surfaces can be found, each in ∞^4 ways, with this same plane section and given four lines. These two surfaces meet further in a space sextic of genus 4.

Two half periods of the space sextic associated with the surfaces C_1, C_2 determine a third associated with a surface C_3 in either the syzygetic or the azygetic way. Two cases are possible.

$$\text{Either} \quad \begin{aligned} C_2 + k_{23}C_3 &= \pi_{23}I, \\ C_3 + k_{31}C_1 &= \pi_{31}I, \\ C_1 + k_{12}C_2 &= \pi_{12}I; \end{aligned} \qquad \text{or} \quad \begin{aligned} k_2C_2 + k_3C_3 &= \pi I, \\ l_3C_3 + l_1C_1 &= \pi I, \\ m_1C_1 + m_2C_2 &= \pi I. \end{aligned}$$

In the first case the three surfaces are not in a pencil; in the second case they are. If the first case occurs the two inscribed four lines must belong to the same half period of C_{23} on π_{23}. Otherwise on C_{23} there would be two distinct half periods isolated but not the third—a lack of symmetry not to be expected. Presumably then the second case occurs when on the plane π the nodal tetrahedra of C_1, C_2, C_3 meet π in inscribed four lines, one from each of the three systems. With respect to the first case we may prove the theorem: (*c*) Given two tetrahedra in general position T_1, T_2, a plane τ can be chosen in four ways such that the edges of T_1, T_2 meet π in the two sets of 6 vertices of two four lines inscribed in a cubic curve

C_{12} and belonging to the same half period on C_{12}. There are two Steiner cubic surfaces C_1, C_2 with nodal tetrahedra T_1, T_2 respectively and the plane section C_{12} such that $k_1C_1 + k_2C_2 = \pi I$. The plane π is the plane of one of the four conics in the pencil of quadric envelopes which touch the 8 planes of T_1 and T_2.

14. *Particular cases of W.*—There are four particular cases of interest in connection with the canonical sextic W each characterized by one condition. When one half period or discriminant factor of the sextic is isolated the others are of two kinds, respectively syzygetic or azygetic with the given one. Thus when the space sextic is determined by a Steiner cubic and the quadric I it acquires a node in one of two ways—either I touches the Steiner surface or I passes through a node of the Steiner surface.

In the case where I touches the Steiner surface at the ordinary point p the space sextic has a node at p. The octavic envelope in which I and the Steiner quartic envelope meet also has the tangent plane at p for a double plane with contact at p whence the ternary quartic C^4 has a node at the same point the W-sextic has an extra node.

In the case where I passes through a node of the Steiner cubic the corresponding line in the plane factors out and the W-sextic is a residual quintic of genus 3 with simple points at the three vertices on this line and nodes at the other three vertices. The curve C^4 however has no singularity.

A third particular case which from a certain point of view is not a special case of the W-sextic occurs when I is a quadric cone and the sets t,τ of generators coincide. Here the web of conics in the plane no longer consists of residual pencils but rather contains an isolated conic and a linearly independent quadratic system of ∞^1 conics. If t is a parameter on the isolated conic the quadratic system cuts this conic in a quadratic system of quadruples determined by a general form $(\alpha t)^4(a\tau)^2 = 0$ with only 8 absolute constants and the locus of the vertices of the diagonal triangles of these quadruples is the W-sextic. The curve C^4 is of hyperelliptic type—the isolated conic doubly covered—with 8 branch points, $(\alpha t)^4$-$(\alpha' t)^4(aa')^2 = 0$. A birationally equivalent form of this sextic is the locus of the 9-th node of sextics with 8 given nodes—a 9-ic curve with triple points at the 8 given nodes. When the 8 points are known this 9-ic can be mapped upon the space sextic in such a way that all of the 120 tritangent planes are rationally known.[13] This case is characterized by the vanishing of an even theta function for the zero argument.

The fourth case—defined by the vanishing of the invariant A of section 11—occurs when the two space cubic curves of section 2 are rational point and line cubic in the same plane. For the space sextic of genus 4 two generators t of I meet the sextic at points crossed by the same three generators τ. For the W-sextic the C^4 is generated by two projective

pencils of conics on coresidual quadruples in such a way that in the projectivity the three degenerate members of the two pencils correspond.

[11] Abstracts I and II, these PROCEEDINGS, **7**, 1921 (245, 334).
[12] Coble, *Trans. Amer. Math. Soc.*, **14**, 1913 (261).
[13] Coble, *Trans. Amer. Math. Soc.*, **17**, 1916 (358).

ON IRREGULARITIES IN THE VELOCITY CURVES OF SPECTROSCOPIC BINARIES

BY HEBER D. CURTIS

ALLEGHENY OBSERVATORY

Read before the Academy, April 24, 1923

The spectrographic velocity curves of several Cepheid variable stars, and of a few other stars not yet definitely known to be variable, show puzzling deviations from simple elliptical orbital motion. These deviations have hitherto been explained by superposing upon a primary elliptical velocity curve a secondary oscillation (generally circular) whose period must be *precisely* one-half or one-third the period of the primary. Such treatment has produced fairly good agreement with the observed velocity data. There is, however, a similarity in the location of the nodes of the primary and secondary curves which savors of artificiality. Moreover, it is very difficult, if not impossible, to devise reasonable and stable multiple systems, or tidally distorted stars, which shall produce such anomalies of one-half or one-third the period of the system.

My failure to construct a dynamically reasonable tidal or other model giving such oscillations in an exact submultiple of the period has led me to the attempt to represent the observational data by means of a purely elliptical velocity curve, plus a *single* oscillation or "hump." This new method of treatment is of considerable interest and, if substantiated by future more accurate spectrographic data, will have an important bearing upon some of the many theories of Cepheid variation, the most puzzling feature of which, as is well known, is the essential synchronism of maximum light with maximum velocity of approach.

Limitations of space make it impossible to give in this paper the curves which have been derived for these stars under the hypothesis of secondaries of a submultiple of the period; these curves will be found in the original papers, references to which are given at the close. In cuts I–VI, shown in Figures 1 and 2, the original spectrographic velocities are plotted without change as given by the respective observers, the dots representing three-prism results, and the open circles those derived from one-prism

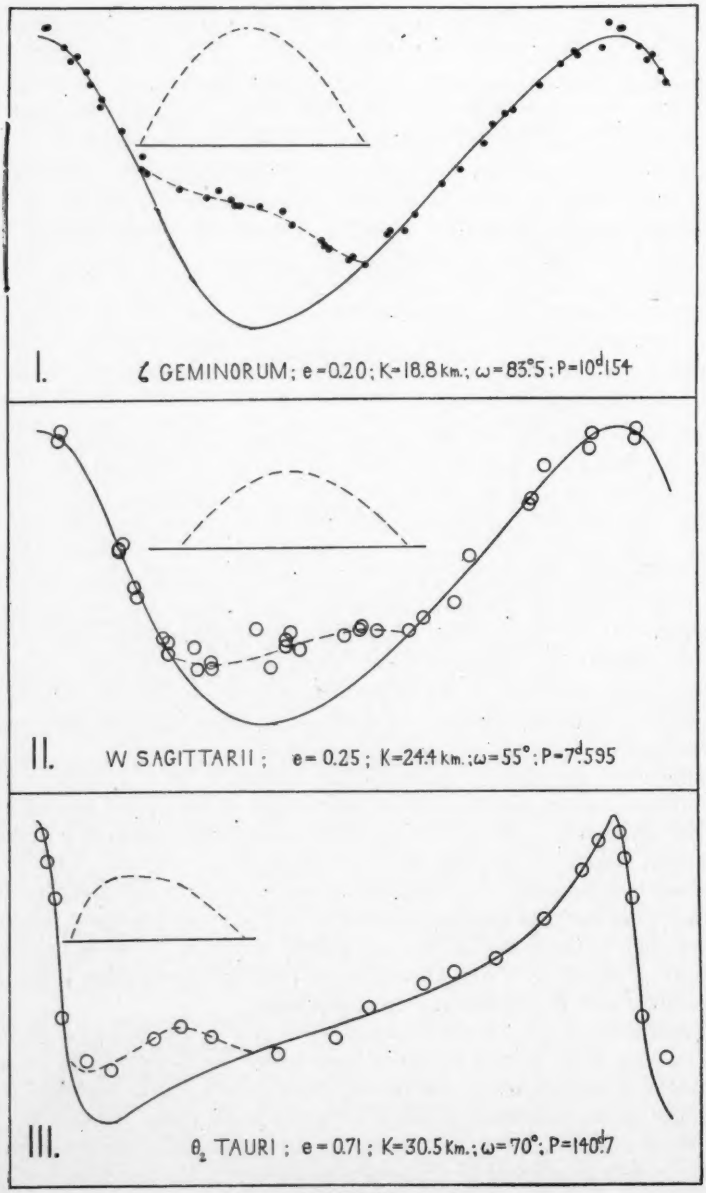
I. ζ GEMINORUM; e = 0.20; K = 18.8 km.; ω = 83°.5; P = 10d.154
II. W SAGITTARII; e = 0.25; K = 24.4 km.; ω = 55°; P = 7d.595
III. θ2 TAURI; e = 0.71; K = 30.5 km.; ω = 70°; P = 140d.7

FIGURE 1

observations. In all the cuts, maximum velocity of approach is the lower peak of the curve, with which the epoch of maximum light in the Cepheids is closely synchronous.

Legends for Fig. 1

I. ζ Geminorum.[1] A Cepheid variable. Observations of high accuracy, obtained by Campbell with the Mills three-prism spectrograph. Recent observations by Mr. Jacobsen at Lick Observatory show the hump in the same place after twenty-three years. The agreement of the present curve, in its undistorted portion, is rather better than from an elliptical orbit with a secondary oscillation of one-third the main period.

II. W. Sagittarii, a Cepheid variable.[2] Observations of moderate accuracy, by Curtiss with the one-prism Lick spectrograph. The agreement in the undistorted portion of the curve is about as good as that secured through a secondary of one-half the main period. It will be noted that the hump is, as before, nearly symmetrical with the velocity minimum.

III. θ_2 Tauri.[3] The observations are fairly accurate, being normal places by Plaskett based on one-prism observations, with the inclusion of a number of three-prism values. The agreement in the undistorted portion of the curve seems better than with a secondary of one-half the period. Not known to be variable.

Though there is a verisimilitude in the humps obtained by this new method of treatment, all occurring close to the time when the star is approaching us most rapidly, an explanation may be as difficult as in the case of the hypothesis of secondaries of an exact submultiple of the period. Two possible explanations may be tentatively given.

1. These humps might be due to a blend phenomenon, for which we have numerous analogies in the curves of a number of spectroscopic binaries. This explanation seems unlikely, first, because blending or line doubling has been sought for in vain in several of the cases treated; and, secondly, the slight but definite asymmetry of these humps is unlike a blend effect.

2. One theory of Cepheid variation, due in its original form to Duncan,[8] explains the synchronism of light maximum and highest velocity of approach, as well as the periodic changes in spectral type, by the assumption that the large and tenuous star whose light we observe is rotating about a darker star in a slightly resisting medium. This medium "brushes back" the rare outer atmosphere of the star on its advancing face. Such a "brushing back" would have the effect of a quasi-depression, most noticeable as the star is coming directly toward us; if the star is large, this quasi-depression may still be convex as viewed from the interior of the star. Such depressions would give rise to "humps" like those drawn in the velocity curves. Their effective depths would be of the order of 1,000,000 km. for ζ Geminorum, and 500,000 km. for W Sagittarii, quantities which are not unreasonable if these stars are large. This explanation is somewhat more difficult in the cases of θ_2 Tauri and φ Persei, for here the quasi-depression would need to have an effective depth of the order of 10,000,000 km. Even this would not be unreasonable were these stars as large as Betelgeuze. Other difficulties in this theory cannot be treated here.

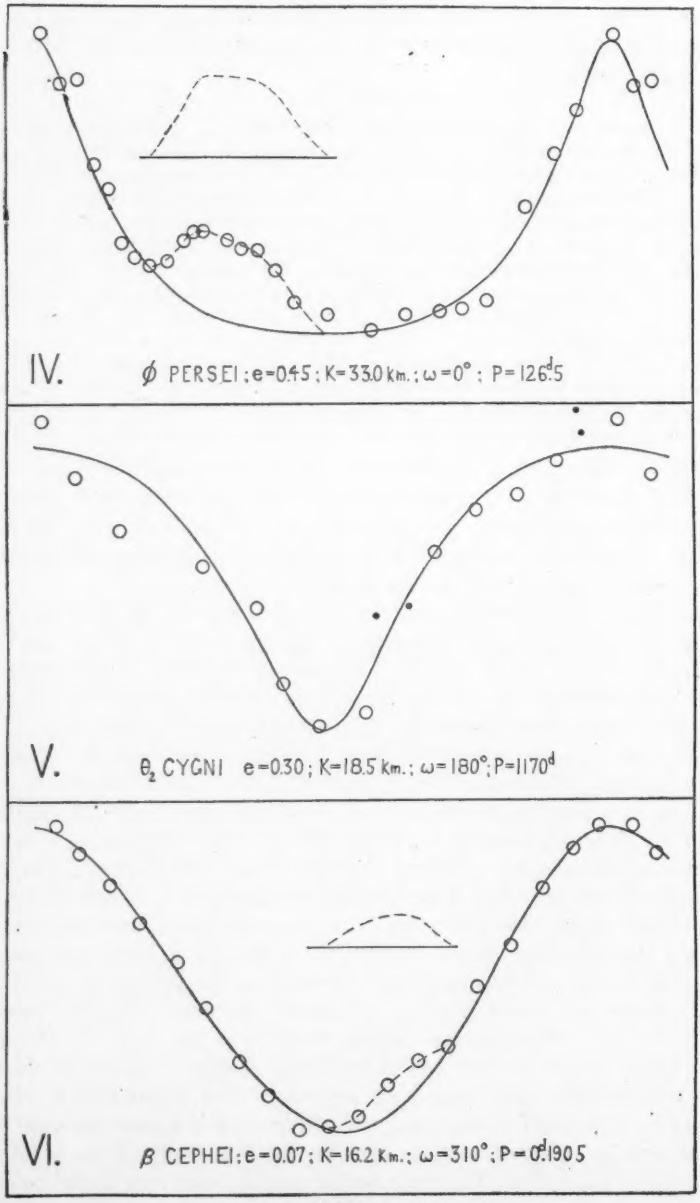
IV.
φ PERSEI; e=0.45; K=33.0 km.; ω=0°; P=126.d5
V.
θ2 CYGNI e=0.30; K=18.5 km.; ω=180°; P=1170d
VI.
β CEPHEI: e=0.07; K=16.2 km.; ω=310°; P=0.d1905

FIGURE 2

Legends for Fig. 2

IV. φ Persei.[4] This star is not known to be a variable. The observations are of moderate accuracy, being normal places from one-prism observations by Jordan at Allegheny. It is a star of very complex and erratic behavior. Cannon[5] (observations not shown) postulated a secondary of one-half the main period, but it is impossible to satisfy Jordan's data in this manner.

V. θ_2 Cygni.[6] Not known to be a variable. The observations are of moderate accuracy, being normal places from one-prism observations by Cannon at Ottawa, with the inclusion of four Lick three-prism values. The observations are grouped almost precisely as would be expected were there a blend effect, but such an effect was searched for without result. Cannon's primary has an eccentricity of 0.18, and he introduces a secondary of one-third the main period. I cannot satisfy these observations with a single hump, as in the cases given before. It is of interest to see that an entirely different orbit from that of Cannon shows some measure of agreement. More observations are urgently needed on this star, particularly just after maximum velocity of recession, to decide on the character and reality of its deviations.

VI. β Cephei.[7] A Cepheid variable of unusually short period $0^d.19$ Crump's observations with the Detroit Observatory one-prism are normal places and of rather high accuracy. My curve does not satisfy these places quite so well as does his orbit, which postulates a secondary of one-third the main period, with a semi-amplitude of but 1.25 km.

One purpose of this paper will be fulfilled if those possessing instruments of adequate power will accumulate many more observations of the light and velocity curves of the Cepheids; only with such assistance will it ever be possible to solve the many puzzles which they present.

[1] Campbell, *Astrophys. J.*, **13**, 1900 (90).
[2] Curtiss, *Lick Obs. Bull.*, **3**, 1904 (62).
[3] Plaskett, *Publ. Dom. Obs.*, **2**, No. 2, 1915.
[4] Jordan, *Publ. Allegh. Obs.*, **3**, 1913 (31).
[5] Cannon, *J. R. A. S. Canada*, **4**, 1910 (195).
[6] Cannon, *Astrophys. J.*, **47**, 1918 (193).
[7] Crump, *Publ. Detroit Obs.*, **2**, 1915 (144).
[8] Duncan, *Lick Obs. Bull.*, **5**, 1908 (82).

ON THE FORM OF THE DISTRIBUTION LAW OF STELLAR VELOCITIES

BY WILLEM J. LUYTEN

HARVARD COLLEGE OBSERVATORY, CAMBRIDGE, MASSACHUSETTS

Communicated April 11, 1923

In the mathematical theory of stellar statistics we need, among others, an analytical expression for the distribution of cosmic velocities. From a theoretical point of view the simplest and most natural form of this "Velocity Law" is the Maxwellian form

$$F(v)dv = \frac{4h^3}{\sqrt{\pi}} e^{-h^2v^2} v^2 dv$$

involving only one constant and assuming a random distribution in direction.

It has been shown by Kapteyn and others that the distribution of the observed velocities does not agree with this assumed expression. In his classical papers on stellar statistics, Schwarzschild assumed the law for the distribution of the observed tangential velocities to be of the form:

$$\psi(T) = 10^{a + c \log T + c (\log T)^2}$$

corresponding to a normal error distribution of the logarithms of the tangential velocities.

The only means we have of verifying the truth of any assumed law of distribution lies in analyzing the motions of all the stars in a given volume of space. With our present scanty knowledge of stellar distances, and of proper motions and radial velocities, especially for the fainter stars in space, this rigorous procedure is not possible. Astronomers therefore have usually contented themselves with analyzing the data for certain groups of stars selected in various ways. On the other hand, we *do* know at present at least a large percentage of all the stars contained in the sphere, with radius of ten parsecs around to Sun as center. According to Kapteyn's latest value for the density, this sphere should contain the Sun and 188 stars. The writer has listed 104 stars (counting double stars as one) for which the trigonometric, spectroscopic and other reliable hypothetical parallaxes point to a value definitely larger than 0.099.

For 83 of these stars radial velocities are known. The space velocities obtained for these same stars were corrected for solar motion. This last vector was computed from the stars themselves by a rigorous solution using Bravais's method, assigning to each star a mass equal to the mean for its spectral class and absolute magnitude.

The resulting solar motion is 25 *km./sec.* directed toward:

$$A = 277°, \quad D = +36°, \text{ or } l = 33°, \quad b = +18°$$

We now have at our disposal 104 tangential velocities and, including that of the Sun, 84 space velocities. The space velocities range from less than 5 *km./sec.* (A. G. CbrE 3 and 36 Doradus) to 319 *km./sec.* (A. G. Lu 5297 = Groomb. 1830).

It was found that the distribution of both space and tangential velocities could be very accurately represented by a function of the Schwarzschild type:

$$\psi(V)\, dV = \frac{h \text{ mod.}}{\sqrt{\pi}} \frac{1}{V} e^{-h^2(\log V - \log V_0)^2} dV.$$

For the space velocities we have

$$h = 1.92, \quad \log V_0 = 1.61, \quad V_0 = 40.8 \; km./sec.$$

V_0 is the geometrical mean of the space velocities. The arithmetical mean, $\overline{V}$, is related to it by the formula

$$\log \overline{V} = \log V_0 + \frac{1}{4h^2 \text{ mod.}}$$

and

$$\overline{V} = 58.5 \; km./sec.$$

Owing to the small number of stars the value of h is as yet very uncertain.

For the tangential velocities uncorrected for solar motion, we have:

$$h = 2.39 \qquad \log T_0 = 1.54$$
$$T_0 = 34.7 \qquad \overline{T} = 43.7 \; km./sec.$$

The accuracy with which these two formulae represent the actual distribution may be judged from figures 1 and 2.

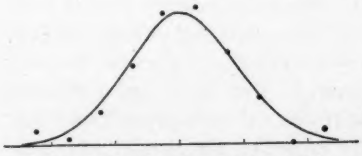

FIGURE 1

Distribution of stellar velocities, taken with reference to the center of gravity.
Abscissae: logarithms of velocities in km./sec.
Ordinates: numbers of stars; highest ordinate corresponds to 21.
Full line: logarithmic error curve.

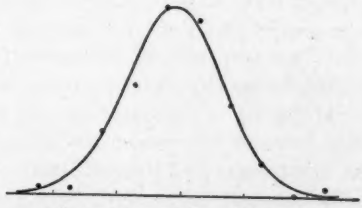

FIGURE 2

Distribution of linear tangential velocities, uncorrected for solar motion.
Abscissae: logarithms of velocities in km./sec.
Ordinates: numbers of stars; highest ordinate corresponds to 29.
Full line: logarithmic error curve.

The question now arises whether the velocities of the missing stars will also follow similar laws. All of these missing stars are probably intrinsically faint and accordingly will, in general, have high velocities.

On the other hand, the fainter stars which have been included consist largely of those which have been selected by parallax observers on account of their high angular velocity. Hence we may expect that, when all stars

nearer than 10 parsecs ultimately become known, the velocities will follow a distribution similar to the one found here.

The evidence from nearby stars, therefore, favors distinctly the assumption that the distribution of the stellar velocities can be represented by a single logarithmic error curve instead of by a combination of two Maxwellian curves or two logarithmic error curves.

THE SOLAR PRELUDE OF AN UNUSUAL WINTER

BY C. G. ABBOT AND COLLEAGUES[1]

ASTROPHYSICAL OBSERVATORY, SMITHSONIAN INSTITUTION

Read before the Academy, April 24, 1923

Does the sun's variation influence the weather? If so, the year 1922 and following months ought to show it.

Referring to the accompanying illustrations, the lower curves give, respectively, the monthly mean "solar constant" values determined by my colleagues and myself at the two Smithsonian Institution observing stations Mount Harqua Hala, Arizona, and Mount Montezuma, Chile, and also the general mean of both stations. These observations cover all months from October, 1920, to September, 1922. The extraordinary drop of solar values during 1922 at once strikes the eye. It is confirmed at both stations.

At the top of the figure we give the mean monthly values for the past four years as determined at Calama, alone, to July, 1920, and thereafter at Montezuma and Harqua Hala. The reader will note prevailingly high solar constant values until 1922, usually exceeding the normal solar constant value of about 1.94 calories, as we have determined it. The curve shows that nothing so outstanding as the change of 1922 to low values has occurred in all that interval. Indeed, nothing so marked has occurred since we began these observations in 1905, so far as our Mount Wilson work can show. That work, however, was fragmentary.

Low solar radiation values continued to prevail in later months of 1922 and early months of 1923, as results yet unpublished will show. A full account of the individual values here summarized may be found in Volume IV of the Annals of the Astrophysical Observatory of the Smithsonian Institution, supplemented by a forthcoming detailed publication to appear as a Supplement of the U. S. Weather Bureau's "Monthly Weather Review." We give the monthly mean values in the accompanying table.

All this being so, has anything unusual occurred in weather conditions which may have been connected with solar changes? We are not to look for anything so simple as a general drop of temperatures all over the world.

Oceans, deserts, mountains, clouds and winds make up too complex a system for such simple reactions. Pronounced departures of some sort from normal conditions, however, we might expect.

It will be recalled that the prevailing characteristic of the weather of the United States for the last couple of years or more is a condition generally

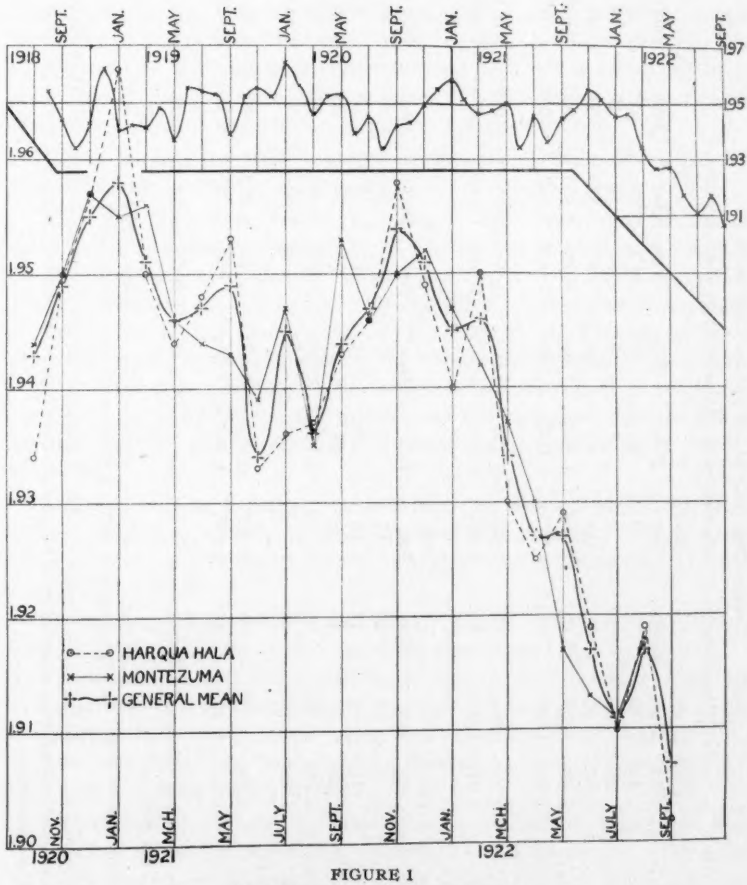

FIGURE 1
Monthly mean values compared

warmer than normal. Thus, from the summary for 1921, U. S. Weather Bureau, Climatological Data, "The outstanding feature of the year was the unusually warm weather in nearly every month. The annual mean temperature was 66.3°, or 2.7° above the normal, making it the warmest

TABLE—MONTHLY MEAN VALUES COMPARED

("Unsatisfactory" values omitted, "Unsatisfactory plus" retained)

	1920			1921								
Month	Oct.	Nov.	Dec.	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.
No. Harqua Hala	11	10	3	9	17	9	15	13	20	5	12	25
No. Montezuma	20	25	21	9	7	13	16	12	17	17	1	5
No. in mean	25	25	22	14	20	18	26	22	23	21	11	27
Mean, Harqua Hala	1.934	1.950	1.957	1.968	1.950	1.944	1.948	1.953	1.933	1.936	1.937	1.943
Mean, Montezuma	1.944	1.950	1.957	1.955	1.956	1.946	1.944	1.943	1.939	1.947	1.935	1.953
Harqua Hala—Montezuma	−0.010	0.000	0.000	0.013	−0.006	−0.002	0.004	0.010	−0.006	−0.011	0.002	−0.010
General mean	1.943	1.949	1.955	1.958	1.951	1.946	1.947	1.949	1.934	1.945	1.936	1.944

	1921			1922								
Month	Oct.	Nov.	Dec.	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.[1]
No. Harqua Hala	19	17	10	13	8	12	17	20	23	18	9	13
No. Montezuma	11	14	12	18	9	11	11	5	10	8	10	2
No. in mean	23	24	15	21	16	20	24	24	24	19	15	13
Mean, Harqua Hala	1.946	1.958	1.949	1.940	1.950	1.930	1.925	1.929	1.919	1.910	1.919	1.902
Mean, Montezuma	1.946	1.950	1.952	1.947	1.942	1.937	1.930	1.917	1.913	1.911	1.918	1.930
Harqua Hala—Montezuma	0.000	0.008	−0.003	−0.007	0.008	−0.007	−0.005	0.012	0.006	−0.001	0.001	−0.028
General mean	1.947	1.954	1.951	1.945	1.946	1.934	1.927	1.927	1.917	1.911	1.917	1.907

Average monthly deviation (Harqua Hala-Montezuma) without regard to sign, 0.0057, or 0.3 per cent. Range of solar variation in monthly means, 2.5 per cent.

General means (Montezuma alone) { August, 1920 Number, 27 Value, 1.930 / September, 1920 Number, 25 Value, 1.942

[1] This month is incomplete. A change in apparatus was made at Harqua Hala after September 20, so that the table closes with that day. Only two satisfactory observations being reported from Montezuma, and these quite out of the general trend, the large difference between the stations should be discounted.

year since the beginning of State-wide records." The monthly reports for 1922, while less pronounced in this sense, indicate warmer than normal conditions on the whole. We start, then, with an excess of heat.

Quoting, now, from Climatological Data: "The record of December, 1922, shows unusual contrasts as to the temperature and precipitation in different parts of the country. In the Southeast it was the warmest or almost the warmest December for 30 years or more, while in the far Northwest it was the coldest December in a like period. The precipitation over Tennessee, Mississippi, and considerable parts of the states adjoining was about as great as ever yet recorded there in December, while an area centering in Kansas had no precipitation, or practically none."

Prof. P. C. Day has kindly sent me in advance of publication summary statements for January and February, 1923, as follows:

"Like the preceding December, January, 1923, was notable for the disturbed atmospheric conditions. Cyclones and anticyclones, mostly well to the northward of their usual tracks, moved in rapid succession across the country. . . . The absence of important anticyclones that usually move into the United States during winter from the far Canadian Northwest was one of the marked features controlling the weather of the month, and the chief factor leading to its moderate character over much of the country. As a result . . . the average sea-level pressure was far below normal over the greater part of the country, and notably so in the far Northwest.

"The outstanding feature of the weather . . . was the almost continuously high temperature . . . over much of the country. At the same time, however, severe winter weather was the rule over New England and much of New York.

"Precipitation occurred with unusual frequency . . . in northern districts west of the Continental Divide and from the Upper Lakes eastward. . . . The snowfall . . . was about average in most higher districts of the far West, save to east and south of the Colorado river, and was heavy in the northern portions of the Lakes region, and especially in the Northeast . . . The average for all the New England States was the greatest in the 35-year period of comparative records.

"The disturbed atmospheric conditions, so persistent during the first two months of the present winter, continued into February . . . The pressure distribution for the month as a whole showed marked variations from the conditions usually expected in February. In the Canadian Northwest Provinces and the adjoining portions of the United States, the averages for the month were far above normal, in fact all sections of the United States and Canada, as far as reports disclose, had averages above normal, a condition rarely experienced.

"The unseasonable warmth which had continued during most of the two preceding months of the winter save over the Northeastern States, terminated with the first few days of February, and the remainder of the month was distinctly cold. . . . The month as a whole was colder than normal in nearly all parts of the country. . . The month's precipitation was less than normal over much more than half of the country."

While it is far too early in the study of the relations of solar radiation and weather to state that the extraordinary solar change caused the unusual winter weather, it does no harm to draw attention to both, in the hope of attracting investigation.

A few remarks on the degree of accordance of our results on the measurement of solar radiation will be of interest.

A comparison has been made between the individual daily values at the two stations. After making a systematic horizontal increase of all Harqua Hala values of approximately 1 per cent to bring the two series onto the same scale, the average accidental difference remaining for 105 days in which satisfactory, or moderately satisfactory, results were had at both stations is only 0.68 per cent. This implies an accidental probable error of 0.41 per cent for a fairly good day's determination at one station. The average deviation of the monthly mean values, 0.3 per cent, would doubtless be smaller than it is if the days observed were always identical at the two stations.

While these values indicate a rather satisfactory degree of accuracy, improvements just made at both stations will probably lead to better work and closer accord beginning January, 1923. Arrangements have been made to carry on the "solar constant" observations daily, when possible, at both stations until July, 1925. It is hoped that by that time it will be possible to make a just estimate of the value of the work for meteorology.

[1] My colleagues, F. E. Fowle, L. B. Aldrich, A. F. Moore, L. H. Abbot, and J. A. Roebling, have each and all contributed so largely in different ways to these results that their names are entitled to co-authorship. It is only to avoid cumbrous citations that they are omitted in the heading.

Only less valuable and indispensable for the research has been the conscientious, painstaking, and enthusiastic work of Messrs. A. Kramer, P. E. Greeley, F. A. Greeley, Mrs. G. M. Bond and Miss M. A. Neill.

We owe, besides, much to the help of the weather bureaus of the United States, Chile, and Argentina, the Chile Exploration Company, and to many citizens of Wenden, Arizona, especially Mr. W. B. Ellison, and Mr. J. E. Matteson.

THE STRATIFIED SUBSIDENCE OF FINE PARTICLES*

BY C. E. MENDENHALL AND MAX MASON

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Communicated April 25, 1923

At least two observers, Brewer[1] and Barus,[2] have recorded the appearance under some circumstances of very marked *stratifications* when fine particles are allowed to subside in water or some other liquid. Under some circumstances, the subsidence takes place in such a way that there is at all times a continuous increase in opacity of the mixture from the top toward the bottom, whereas with stratified subsidence, the mixture divides itself into sections each of more or less uniform opacity, but with the opacity of succeeding sections increasing by sudden jumps from the top down. Barus began a minute study of the conditions under which stratified subsidence occurred, paying particular attention to the rate of fall of the boundary surfaces as affected by the total amount of suspended material (in his case, tripoli), or by the presence of a dissolved substance (salt, alcohol, or ether), but arrived at no conclusion as to the cause of stratification. Brewer suggests that the strata may be due to the formation of a series of weak compounds of water with the suspended material.

Our attention was called to the problem by Professor Warren J. Mead, who observed marked stratification with suspensions in water of rock of the Cucuracha[3] formation from the Panama Canal, the rock having been kept moist and then ground up with water in a mortar. There being an abundance of this material available, we have done most of our work with it, though powdered quartz and glass also have been used. In common with the previous observers, the suspensions were placed in glass tubes. We were early led to place a number of our tubes in an inside dark room in order to secure more uniform and constant temperature conditions. It was found that, if stratifications had already developed in the tubes, they disappeared after some hours in the constant temperature room, and that tubes of freshly suspended material would not develop stratifications in the constant temperature surroundings. Tubes placed near a window in an outside room developed layers, but similar tubes on a slowly rotated turn-table did not develop layers.

These observations suggested a series of experiments, which need not be considered in detail, which led to the conclusion that *stratification only occurs when there is a lateral temperature gradient* across the liquid. This idea was confirmed by observations with various specially controlled temperature gradients. Rectangular tubes were made, two opposite sides of which were in contact with water baths of different constant temperatures, and radial gradients were produced by an arrangement of three

concentric tubes, the central one carrying a current of warm water, the outer a cool bath. A considerable number of tubes were prepared, varying in diameter from 5 to 40 mm. and in length from 30 to 100 cm., and with these we attempted to determine more carefully the positions and velocities of the boundary surfaces as affected by the various factors, the most important of these being the magnitude of the temperature gradient, the amount of suspended material, the length of time during which a tube was allowed to stand after shaking before layers were allowed to form, and the diameter and thickness of wall of the tubes.

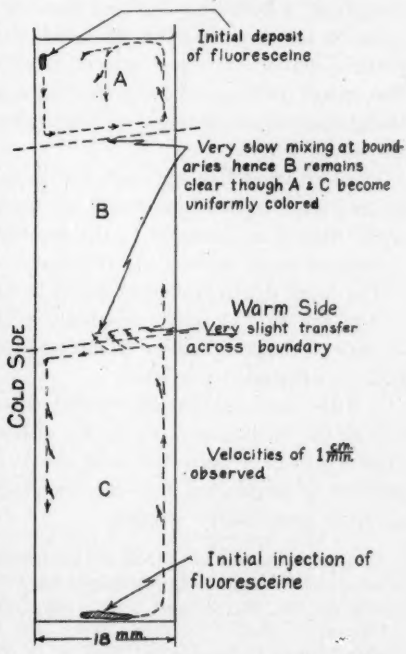
The effect of allowing a tube to settle under uniform temperature conditions (no layers) is not only to reduce the total amount of suspended material, but to change the distribution of the particles through the height of the tube, the larger particles, of course, settling more rapidly. The distribution of suspended material in an unstratified tube at two different times after mixing may be represented by curve of the form given in Fig. (4) (A and B). It is to be expected that the effect of a given temperature gradient will be different according to the length of the preliminary settling of "aging" interval.

With a given initial amount of suspended material, if a temperature gradient is produced immediately after shaking up, the first result is the appearance of boundary surfaces near the top of the tube, whereas if more time is allowed for "aging," the boundaries first appear farther apart and distributed through a greater length of the tube. Considering these results in connection with the curves mentioned above which show the change of distribution of particles upon aging, it is evident that the strata appear in that part of the tubes where the amount of suspended material increases appreciably with increasing distance downward from the surface. If tubes are allowed to settle for a sufficient time (some weeks or months) stratifications may be produced by temperature conditions (such as those in our constant temperature room) too uniform to produce strata in the initial, more concentrated, suspension. On the other hand, with a given amount and distribution of suspended material, the boundaries will in general be farther apart the larger the temperature gradient, and a temperature gradient may easily be obtained so great that no strata will be produced, or strata already produced will be destroyed. From all this, we conclude that both a temperature and a density gradient are necessary to produce stratification, and that the temperature conditions must bear, within certain limits, proper relation to the density distribution.

Temperature gradients such as we have been discussing will, in general, produce convection currents, and these we consider to be the direct cause of stratification. We were able to demonstrate the existence of a complicated system of convection currents in a stratified tube by carefully injecting at various points small amounts of water colored with fluorescein,

care being taken that the injected water was at the same temperature as that in the tube. Fig. (1) shows the convection system. There is but a very slow mixing of the liquid in adjacent compartments across the boundary, as shown.

The general process of stratification is probably somewhat as follows: Suppose a tube has been aged at constant and uniform temperature so that settling without stratification has occurred. The particles in suspension being distributed continuously in size, the macroscopic density of the aged mixture increases continuously towards the bottom. Suppose now that the temperature of one side of the tube be lowered so that a lateral difference of temperature is established across the tube at all levels. If no vertical density variation were present, the liquid would be set into convection, the entire volume of the liquid furnishing a single continuous convective pattern, with lines of flow near the wall which would extend continuously from top to bottom on the cold side and from bottom to top on the hot side. If a sufficient density gradient due to suspended particles be initially present in the liquid, however, a different type of convection will be formed. A portion of the liquid near the cold wall becomes denser than the average at its depth and flows downwards. As it falls, its negative buoyancy due to temperature difference becomes continuously less since the density of the liquid into which the portion is flowing is greater on account of its larger content of suspended particles. Reaching a level where its temperature contraction balances the excess density of the main mass of fluid at that depth it is in stable equilibrium, and the downward flow ceases. Fluid elements near the cold wall thus tend to flow a definite distance downwards.



CONVECTION SYSTEM IN LAYERS
FIGURE 1

As the uppermost element A moves down, fluid from the top of the column follows in its wake, so that while the elements below A show variation

in density due to particle content, the new elements which are continuously brought to the top of the cold wall are drawn from the top portion of the liquid and are of nearly uniform density.

When A reaches its equilibrium depth, its downward flow is decelerated and it turns into the interior of the liquid, this motion being followed by the elements behind A. The equilibrium depth for A thus acts approximately as a boundary limiting the flow, and a local convection is established in the upper portion of liquid, this portion being mixed so that its particle content becomes uniform, equal to the average content of the portion before mixing. Thus a definite upper layer is formed, with an internal convection system, the density being uniform and less than that of the liquid below.

The bottom of this layer acts for the fluid below it, as did the top surface of the liquid in the formation of the top layer. A second layer establishes itself, similar in character to the top layer, and in this manner the entire volume of liquid is divided into layers.

The layer depth is characterized by the fact that the thermal contraction of the fluid elements originally at its top is sufficient to increase the density to that originally obtaining at its bottom, due to the increased content of suspended particles.

A little consideration shows that this conception of the role of convection in the formation of strata fits at least qualitatively all of the observations which have been described above, namely, the effect of changing the amount of suspended matter, changing the temperature gradients, and aging or preliminary settling.

* A preliminary account of this and the following paper was presented to the National Academy of Sciences at its meeting in April, 1921, by C. E. Mendenhall.

[1] Brewer, W. H., *Memoirs, Nat. Acad. Sci.*, Vol. III, p. 165.

[2] Barus, C., *Bull. U. S. Geol. Survey*, Vol. V, p. 515.

[3] See Donald F. MacDonald, *Bull. U. S. Bur. Mines*, No. 86, 1915.

THEORY OF THE SETTLING OF FINE PARTICLES

By Max Mason and C. E. Mendenhall

Department of Physics, University of Wisconsin

Communicated April 25, 1923

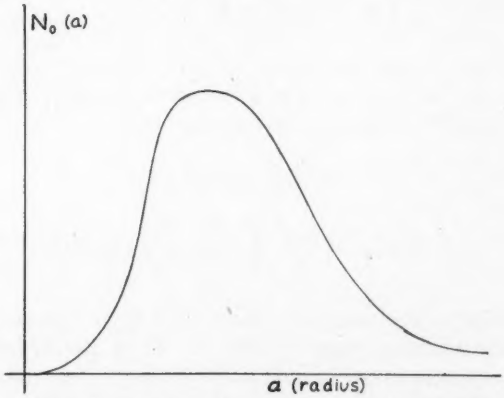
An attempt was made to connect quantitatively the various conditions with the position of layers formed after aging, and with the velocity of fall of a boundary surface.

Suppose the tube contains initially a uniform distribution of suspended particles of various sizes, and consider for simplicity these particles as

being spheres of the same density. Describe the frequency of occurrence of spheres of different radius, a, by the function $N_0(a)$, so that initially the number of spheres per cubic centimeter in the range $(a, a + da)$, is given by $N_1(a)\,da$ (Fig. 2). Let the tube be maintained at a uniform and constant temperature during the time t. During this time the particles of different size fall at different velocities, V, determined by Stokes' Law of fall,

$$V = \beta a^2, \qquad \text{where } \beta = 2g\,\delta/9\mu$$

In this equation, δ is the excess density of particle over the density of water, g is the acceleration of gravity and μ the viscosity of the water.



Number of particles as function of radius

FIGURE 2

After settling for time t, the part of the liquid above depth x will be free of all particles larger than those which fall through a distance x in time t, *i.e.*, all particles for which $V \geqq x/t$ or $a^2 \geqq x/\beta t$. Then it is evident that the total number per cc. at depth x at time t is[1]

$$\int_0^{\sqrt{x/\beta t}} N_0(a)\,da,$$

and the macroscopic density of the mixture is, taking the density of the water as unity, $1 + D(x, t)$ where

$$D(x, t) = \frac{4\pi\delta}{3} \int_0^{\sqrt{x/\beta t}} a^3 N_0(a)\,da.$$

Let us assume that through cooling the density is increased by the factor $1 + \epsilon$. Then if t is the age of settling, the bottom of the first layer will be formed at a depth x_1, such that

$$[1 + D\ (o, t)]\ [1 + \epsilon] = 1 + \mathrm{D}\ (X_1, t),$$

and in general, if x_i is the top of the ith layer,

$$[1 + D\ (x_i, t)]\ [1 + \epsilon] = 1 + D\ (x_{i+1}, t).$$

Now both ϵ and D are small compared to unity, so that

$$\epsilon + D\ (x_i, t) = D\ (x_{i+1}, t).$$

Since $D\ (o, t) = o$, we have $D\ (x_1, t) = \epsilon$, $D\ (x_2, t) = 2\epsilon$, etc. *The depths x_i, x_{i+1} of the boundaries of the ith layer formed after settling for a time t, are therefore conditioned by the equation*

$$D\ (x_i, t) = i\epsilon,$$

where

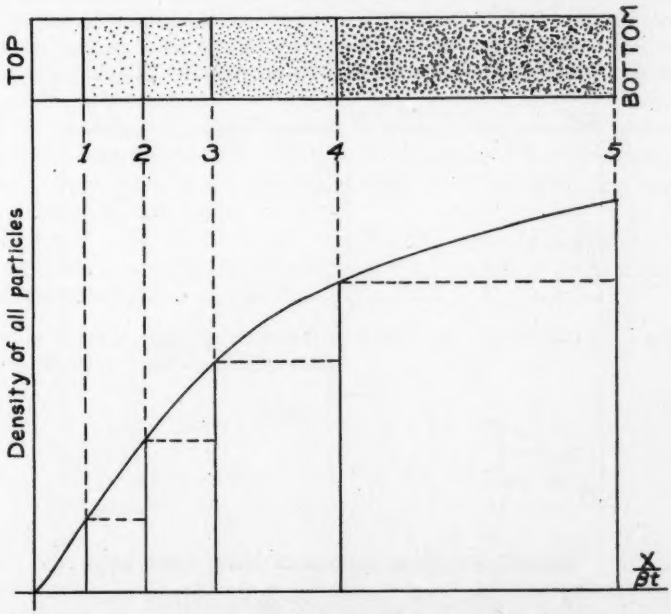
$$D(x, t) = \frac{4\pi\delta}{3} \int_0^{\sqrt{x/\beta t}} a^3 N_0\ (a)\ da.$$

The equation conditioning the position of layer boundaries has the graphical interpretation given in Fig. 3. Erect the ordinates to the curve equal in length to the integral multiples of ϵ. Their position on the axis of $x/\beta t$ gives the values $x_i/\beta t$.

It is seen that the layer thicknesses increase with ϵ and decrease with δ. The effect of multiplying $N_o\ (a)$ and hence $D\ (x, t)$ by a constant (increasing the concentration of suspended particles) is to decrease the layer thickness. If temperature gradients and concentration be increased in the same ratio, there will be no change in layer position. Since D depends on x and t only through their ratio, it follows that the depth of a boundary is directly proportional to the time of settling before layering. The curves of Figure 4 give the density as a function of depth for two different times of aging, one four times as great as the other. The vertical lines show the positions at which layer boundaries should form if two tubes aged by these amounts were subjected to the same transverse temperature gradient.

In attempting to check these conclusions quantitatively by more careful measurements of the position at which layers formed, and of their velocity of fall, we have been hampered by the difficulty of identifying and controlling all of the variables which enter into the problem. We were especially interested in determining the relation of the magnitude of the

temperature gradient (*i. e.*, the violence of the convection) to the position and rate of fall of a boundary. For this purpose we mounted, in the constant temperature room, two groups of tubes, as nearly identical as possible and filled with the same mixture, one group being twice as far as the other from an incandescent lamp (say 500 watts) whose radiation produced the necessary temperature gradient. Under these circumstances, in spite of the care taken to have the several tubes as alike as possible in size and



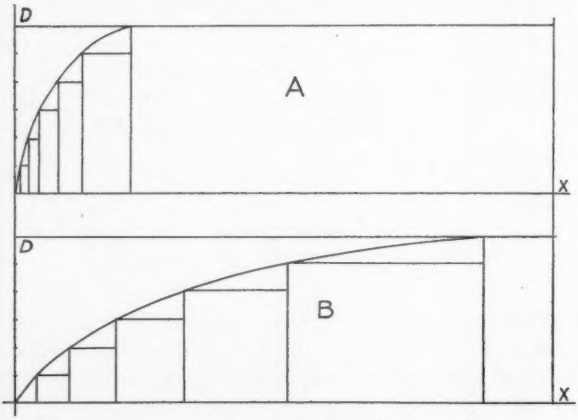
LAYERS DETERMINED BY VARIATION OF DENSITY.
Boundaries of Layers numbered

FIGURE 3

exposure, there was always a considerable variation among the tubes of each group as to the position at which boundaries appeared, and in their rate of fall. Many repetitions of this experiment showed that these variations were accidental, in the sense that they were not characteristic of particular tubes or exposures, nor associated with any other variable which we could identify. Similar results were obtained with tubes in water baths. These irregularities rendered useless our attempts to determine the distribution function N_0 by observations of the position and velocity of boundaries, but we were able to show that the rate of fall of boundaries is in-

dependent of the temperature gradient within quite wide limits, and that the position of layers as a function of concentration, time of aging, and thermal gradient were qualitatively in accord with the theoretical predictions.

Two special cases should be mentioned. In one a convection was maintained by surrounding the upper part of a tube with a cold bath, the lower part with a warm bath. A single sharp boundary was formed, the liquid above being practically clear and that below only slightly opaque, and this boundary remained practically stationary for weeks. This boundary is no doubt the division between two convective patterns formed in the following manner. In the upper portion of the tube, which is in the cold bath, the liquid next the walls is cooler than the average at the same



Curves showing density of liquid after aging (A) for time t_1, (B) for time $4t_1$

FIGURE 4

depth and hence there is the tendency to produce a convection down the walls and upwards in the interior. In the lower portion of the tube, which is in the warm bath, the tendency to convection is opposite—up the walls and down in the interior. The temperatures in the two baths were quite different, so that the convections were rapid, that in the lower portion being sufficiently rapid to scour the particles from the bottom and keep them stirred up in the part under the boundary, which thus was maintained in a turbid state.

The results of our observations may be summarized as follows:

Stratified subsidence occurs only when convection currents are present.

The velocity of a given boundary decreases slightly as it falls, especially during the early part of its fall.

All boundaries fall approximately as if they started from the top at the time settling began.

Other things being equal, a small temperature difference (weak illumination) will produce more strata than a large one in similar mixtures; with the same temperature differences more layers are produced in a more concentrated mixture.

The rate of fall of a boundary is determined primarily by the rate of fall of the particles, independent of convection, that is, with a given mixture it will depend primarily upon the distribution of particles of different size in the region where the boundary is first formed.

A sudden change in the intensity of illumination will result in a sudden shift of all the boundaries.

Taking opacity as a rough measure of amount of suspended material per cc., it is found that the distribution of this material in a layer may be almost exactly uniform, but in general tends to be denser in the upper portion than in the lower.

From these we draw the conclusion that convection currents are the primary cause of stratified subsidence, and offer a qualitative explanation of the mechanism of the process.

[1] Dispersion due to Brownian movement is here neglected. In the cases investigated, this assumption was justifiable.

THE QUASI EQUATION P = TdV/dT

BY EDWIN H. HALL

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Read before the Academy, April 24, 1923

Let A, figure 1, be a very large plate of some metal and B a like plate of a different metal. From each plate runs a wire of the same material as the plate and these two wires join at j. The loop in the A wire indicates the

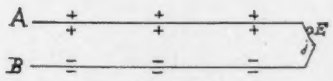

FIGURE 1

windings of an electromagnetic engine which can, upon occasion, introduce an electromotive force tending to send current from B to A or vice versa. When E is inactive, the two plates are, because of the Volta effect, in a state of charge indicated by the plus and minus signs of the figure. The

Volta contact difference of potential, or "Volta effect," is defined as the difference of potential, V, of any two points, one at the free surface of A, the other at the free surface of B.

Let us suppose that now the engine E is set into action so as to produce an e.m.f. tending to send a positive current from A to B. This impressed e.m.f. may be as small as we please, infinitesimal compared with V, but if the plates are, as we suppose, very large indeed, unit quantity of positive electricity will, if sufficient time be given, pass from plate A to plate B, thus decreasing the positive charge on A by unity and decreasing the negative charge on B by unity, without making any appreciable change in the difference of potential of the two plates or of any two points just at the free surfaces of the two plates respectively.

Now it seems to be generally admitted, on thermodynamic grounds, that in this operation, which occurs at the constant temperature T,

$$\textit{the heat absorbed} = T\frac{dV_{AB}}{dT},$$

where V_{AB} = (*Volta potential of* B) — (*Volta potential of* A), but there is difference of opinion as to the amount of this absorbed heat.

Evidently it includes the Peltier effect heat, P_{AB}, at the junction j, and, if we assume there is no other heat absorption, positive or negative, we get the relation

$$P_{AB} = T\frac{dV_{AB}}{dT} \qquad (1)$$

Bridgman[1] has given us the history of this formula. It was first published by Lorentz, in 1889, but remained comparatively unnoticed and was re-derived by Kelvin eight or ten years later. Lorentz, however, accepted the suggestion of Budde that there may be an absorption or evolution of heat at the free surface of the metals, and Kelvin, after finding that the formula, when tested by the experimental data available, "failed by a thousand fold," admitted the same possibility. Richardson,[2] however, by a new course of argument arrived at the formula as written, and K. T. Compton,[3] for the purpose of testing its correctness, undertook measurements of the "temperature coefficient of contact potential," $(dV \div dT)$, between nickel, which we will call metal A, and iron, which we will call B. According to his experiments this temperature coefficient, in the neighborhood of 40° C., is about 0.00165 volt per degree,—that is, about fifty times as large as it should be if equation (1) is correct. Compton does not, however, regard this test as conclusive, and he remarks, quite justly, that measurements in extremely high vacua can alone settle the question here at issue.

Meanwhile, it may be remembered that both Bridgman[1] and I[4] have called in question the soundness of Richardson's argument deriving (1), and that I have made the specific suggestion that a surface thermal effect, such as Lorentz and Kelvin admitted the possibility of, can be accounted for by an application of the mass-law of chemical equilibrium to the relations of the free electrons and the positive ions of the metals. I now propose to apply this suggestion quantitatively in the case of nickel and iron.

The following passage is taken from the first of my two papers just referred to in a footnote:

"Let n_e be the number of free electrons per unit volume of a metal and n_i the number of positive ions, atoms lacking each an electron. Within the interior of a metal in electrical equilibrium we must have $n_e = n_i$. At the surface, if the metal has a static charge, we do not have such equality. Now, according to my view, the free electrons and the positive ions, which are the products of the ionization of the atoms, obey the mass-law, so that everywhere in a metal we have

$$n_e \times n_i = a\ constant.$$

"If this view is sound, when we take electrons from one plate of a condenser and convey them to the other plate, we thereby disturb the electrical equilibrium in each metal. If one metal loses, for example, q electrons, we cannot have mass-law equilibrium in this metal until sufficient new ionization occurs therein to make the total number of free electrons in this metal only $\frac{1}{2}\ q$ less, and the total number of positive ions $\frac{1}{2}\ q$ more, than at first. In the other metal the converse operation must take place, re-association occurring there until the number of free electrons is $\frac{1}{2}\ q$ more and the number of positive ions $\frac{1}{2}\ q$ less than before. These processes of ionization and of re-association would balance each other in heat production if the two plates of the condenser were of the same metal, but otherwise they do not. Of course, wherever in the metals the ionization and the re-association primarily occur, the result will presently appear at the surface, since any excess or deficiency of electrons in the interior of a metal must correct itself at the expense of the surface. Practically, then, we may regard the ionization and the re-association as occurring at the surface only."

Now what is called, in the language of ordinary convention, the passage of unit quantity of positive electricity from A to B, is really the passage of $(1 \div e)$ negative electrons from B to A,—that is, in our case, from iron to nickel,—and this operation will, according to the passage quoted above, involve the releasing of $(1 \div 2\ e)$ electrons from atomic unions at the surface of the iron and the passing of $(1 \div 2\ e)$ electrons into atomic unions at the surface of the nickel. According to data already published,[5] as derived from the theory of thermo-electric action, etc., which I have been developing for several years, the ionizing potential at 40° C. is about 0.398 volt within iron and about 0.144 volt within nickel. Accordingly, freeing $(1 \div 2\ e)$ electrons in iron and re-associating $(1 \div 2\ e)$ electrons in nickel would absorb an amount of heat equivalent to the work of raising $(1 \div e)$ electrons through a potential lift of $(0.398 - 0.144) \div 2$, or 0.127, volt.

According to Bridgman's[6] experiments, the Peltier effect heat in this

case, the heat absorbed by $(1 \div e)$ electrons in going across the junction j, figure 1, from iron to nickel, is, at 40° C., equivalent to the work of raising these electrons through a potential lift of about 0.011 volt. This, added to the heat effect dealt with above, would give the equivalent of 0.138 volt potential lift.

On the other hand, according to Compton's experiments referred to above, we have in this case

$$T \frac{dV_{AB}}{dT} = 313 \times 0.00165 = 0.516 \text{ volt.}$$

So, if we use, following Bridgman,[1] the equation

$$P_{AB'} = T \frac{dV_{AB}}{dT}, \qquad (2)$$

meaning by $P_{AB'}$ the heat energy absorbed at j *plus the heat energy absorbed at the free surface of the metals*, we find that, with my interpretation of the surface heat effect, and with the data at hand, the right-hand member of the equation, 0.516, is about four times the left-hand member, 0.138.

This outcome is, of course, to be regarded as satisfactory only in comparison with the test of equation (1), in which the right-hand member comes out about fifty times as large as the left-hand member. Professor Comptom thinks the ratio fifty can perhaps be accounted for by errors of experiment, but to account for the ratio four would seem to be a smaller task.

In fact, perhaps we need not depend very much here on the assumption of experimental errors. The ionization potentials used above for iron and nickel are the values that hold, according to my reckoning, well within the body of the metals. These values are much smaller than the values found by experiment in the case of metallic vapors, and it may well be that in the very surface film of a metal the ionization potential is decidedly larger than it is farther within. If we should multiply by four the ionization potentials which I have used for iron and nickel, respectively, the two sides of (2) would be almost exactly equal. This, however, is merely a suggestion.

It is worth while to inquire how great an excess or deficiency of electrons per unit area of the surface of a piece of metal may be required to produce a given negative or positive potential. If we, for example, consider a sphere 5 cm. in radius, the density, electrostatic, of surface charge required to make a difference of one electrostatic unit, 300 volts, in the potential of the sphere is

$$\rho = r \div 4\pi r^2 = \frac{1}{20\pi} = 0.016.$$

This is about 3×10^7 electrons per sq. cm. Now the number of atoms exposed in 1 sq. cm. of a copper surface is about 2×10^{15}. So, on a sphere of copper 5 cm. in radius, an excess of one electron for each 7×10^7 surface atoms will charge the sphere to a negative potential of about 300 volts.

Note: If the "ionization potential" is really considerably greater near the surface of a metal than at points deeper within, the degree of ionization,—that is, the number of free electrons and of positive ions per cu. cm.,—is probably less in the surface layer than in the interior. Accordingly, we should expect the specific conductivity of the surface layer to be smaller than that of interior layers, and the specific conductivity of thin films to be less than that of thicker pieces, which experiment shows to be the case. It seems doubtful, on the other hand, whether any practicable degree of surface charge would affect appreciably the conductivity of even a very thin piece of metal.

[1] *Physic. Rev.*, N. S., **14**, pp. 306–347, October, 1919.
[2] *The Emission of Electricity from Hot Bodies*, Longmans, Green & Co., 1916.
[3] *Physic. Rev.*, N. S., **7**, pp. 209–214, February, 1916.
[4] These PROCEEDINGS, **4**, January, 1918, and **5**, June, 1919.
[5] *Ibid.*, **7**, pp. 98–107, March 1921.
[6] *Proc. Amer. Acad. Arts Sci. Boston*, **53**, 269–386. March, 1918.

JOINING THE INFRA-RED AND ELECTRIC WAVE SPECTRA

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NELA RESEARCH LABORATORIES, NATIONAL LAMP WORKS

Communicated May 8, 1923

Historical.—In the first decade following Hertz's[1] discovery of electric waves in 1888 and his success in repeating a number of classical optical experiments with waves 60 cms. long, progress towards shorter and shorter electric waves went rapidly forward. Lebedew[2] obtained and experimented with waves which he estimated at 6 mms., and Lampa[3] worked with wave-lengths believed to be 4 mms. In 1918 Möbius[4] described some short wave experiments of his own and reviewed Lebedew's and Lampa's work, arriving at the conclusion that both had underestimated the wave-lengths with which they were dealing and gave 1 cm. as a more probable value for Lebedew's waves and 7 mms. for Lampa's. Möbius himself was unable to get waves shorter than 7 mms. He obtained evidence of ripples of shorter equivalent wave-lengths but no shorter regular waves. His paper gives a very fair idea of the difficulties he encountered.

Contemporaneous with this advance toward shorter electric waves,

there was a corresponding progress in the infra-red spectrum toward longer waves. Following Langley's classical researches in the solar spectrum, Rubens[5] and Paschen[6] separately pushed the region of accurate wave-length determination to 9.4μ, sixteen times the wave-length of yellow light. In 1897 Rubens and Nichols[7] devised the method of multiple reflection or "reststrahlen" by which the known boundaries of the infra-red spectrum were extended tenfold. A further extension was obtained by Rubens' and Wood's[8] method of focal isolation. By the aid of this latest method and by using a quartz mercury arc as a source, Rubens and Von Baeyer[9] were able in 1911 to obtain and measure heat waves 0.320 mm. in length. Comparing the number of skilled experimenters, better instruments, and more rigorous methods of work in the infra-red spectrum with those previously employed in short electric wave experiments, it seemed that a systematic attack to bridge the gulf of missing wave-lengths between the two spectra had better chances of success if made from the less developed or electric wave side.

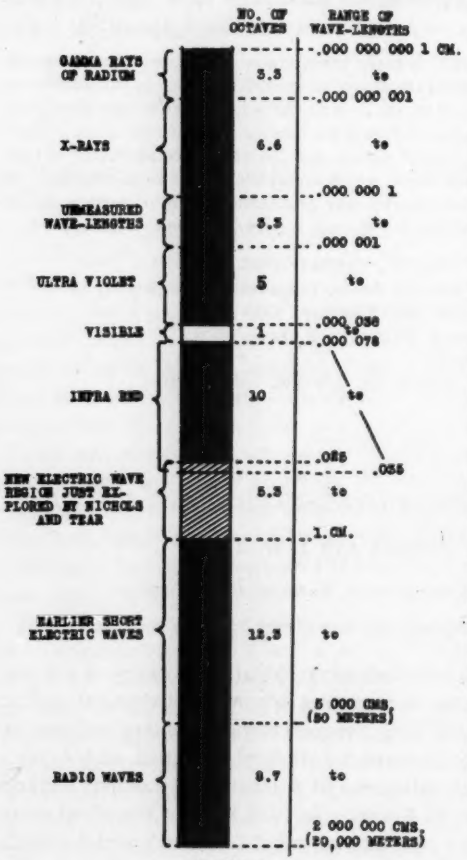

FIGURE 1

Chart of the complete electric wave spectrum

Instruments and Methods. —At the outset of our experiments it was plain that until instruments were devised of a different order of sensitiveness and efficiency from any heretofore used and until radical improvements in methods of experimentation were perfected, no certain advancement toward shorter electric waves could be hoped for. As the instruments and methods employed in the present experiments will be fully described

in a series of two or more papers to appear later in the *Physical Review*, only the briefest reference is needed here.

(1) The electric wave receiver was a modified form of the Nichols radiometer.[10] When electric waves fall on a conductor they cause oscillating currents of the same frequency in its surface layers. Part of the energy of these oscillating currents is converted into heat by the electrical resistance of the conductor, and the conductor is in consequence very slightly warmed. To make use of this phenomenon in an electric wave receiver, the usual blackened vanes of the Nichols radiometer suspension were replaced by heater elements made of very thin deposits of bright metallic platinum on thin mica strips. These heater elements were shielded on one side and so disposed about the rotation axis of the suspension that when warmed by the action of electric waves the radiometric forces on the two vanes combined to produce rotation of the receiver system. This type of receiver proved to be a hundred or more times as sensitive as any quantitative receiver previously used for short electric waves.

(2) The oscillator was in principle the familiar Hertzian doublet with minute tungsten cylinders substituted for platinum. It also embodied a number of other improvements and accessories.

(3) In addition to a Boltzmann and a Fabry and Perot interferometer, a wholly new form of reflecting echelon analyzer was employed in wave-length measurements. The advantage of the new instrument was that in addition to accurate wave-length determinations, it gave the approximate wave form as well.

(4) One of the principal improvements in method was the use of a second identical receiver to which a fraction of the beam of radiation emitted by the oscillator was deflected. This second receiver served as a check on the performance of the oscillator, and errors due to irregularities of emission could thus be eliminated from the results.

Results.—By the patient development and use of the foregoing equipment and procedure, shorter and shorter electric waves were isolated and measured from Lampa's and Möbius' limit of 7 down to waves 0.220 mm. in length. We have thus found it possible to manufacture electric waves shorter than the longest known waves emitted by matter at high temperatures.

We further used our electric wave receiver in two different forms to remeasure Rubens' and Von Baeyer's 0.320 mm. heat waves obtaining results identical with those recorded by these earlier investigators.

The present methods and results throw open for intensive study practically the last unexplored region in the whole extent of the electric wave spectrum which spans the enormous interval from long radio waves to the shortest gamma rays of radium, a wave-length ratio of 20 million billion to one.

Discussion.—The more feasible method of charting the whole sweep of this comprehensive electric wave spectrum is by using geometric intervals such as the ascending powers of two used in laying off a piano keyboard on which the wave-lengths of two notes an octave apart are to each other as two to one. Such a spectral chart is shown in figure 1. On the left are the usual names given to the different main divisions of the spectrum. To these, braces are attached to show their extent on the black strip representing the spectrum. Beyond is shown the number of octaves embraced in each division and furthest to the right the actual wave-lengths roughly corresponding to these regional boundaries. Of the 55 octaves of the spectrum, shown in figure 1, only one octave contains wave-lengths visible to the eye, and there is but one remaining region of 3.3 octaves between the ultra violet and X-ray spectra in which wave-length measurements have not been made. Waves lying in this interval have been observed, and quantum relationships give a sound theoretical basis for calculating these wave-lengths, but no interference nor defraction measurements have thus far been carried out in this limited region.

Matter under the action of heat is capable of giving off radiations in the so-called infra-red, visible, and ultra violet spectra; gamma rays are the natural accompaniment of radioactive disintegration, and there are various static electric phenomena in the atmosphere giving rise to pulse-like disturbances resembling fragments of very long electric waves. But X-rays and the old and new short electric waves we may still regard as artificial or purely products of laboratory manufacture.

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